Scaling Properties of Simple Limiter Control

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"Simple limiter control" of chaotic systems is analytically and numerically investigated, proceeding from the one-dimensional case to higher dimensions. The properties of the control method are fully described by the one-parameter one-dimensional flat-top map family, implying that orbits are stabilized in exponential time, independent of the periodicity and without the need for targeting. Fine-tuning of the control is limited by superexponential scaling in the control space, where orbits of the uncontrolled system are obtained for a set of zero Lebesgue measure. In higher dimensions, simple limiter control is a highly efficient control method, provided that the proper limiter form and placement are chosen.

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Chaos is composed of an infinite number of unstable periodic orbits of diverging periodicities. In order to exploit this reservoir of characteristic system behavior, methods to stabilize (or "control") such orbits using only small control signals have been developed [1–4]. Practical applications often require that the orbits be quickly targeted and stabilized. As an example, the use of unstable orbits for signal transmission in telecommunications would demand a very fast computation of the control signal, as the signal frequency is in the GHz range. In biology, where control of low-dimensional chaotic firing of neurons [5] is a potential candidate for cortical information encoding, a very efficient control mechanism is required as well. This is implied by a comparison between typical cortical reaction and neuronal interfiring times (~ 100 ms vs ~ 20 ms).

For the classical Ott-Grebogi-Yorke [1] and for feedback control, this is a problem. Recently, Corron and coworkers [6,7] introduced a new control approach (termed "control by simple limiters") and suggested that it could overcome the limitations of the previous methods. The general procedure can be summarized as follows: An external load is added to the system, which limits the phase space that can be explored. As a result, orbits with points in the forbidden area are eliminated. The authors also observed that modified systems tend to replace previously chaotic with periodic behavior. The authors tested their approach successfully in different experimental settings, but gave very little explanation of the phenomenon. In our contribution, we provide the theoretical analysis of the method, and we uncover underlying principles that are of relevance for applications. In particular, we address the following main issues: (1) Localization of orbits of prechosen periodicity. (2) Convergence properties of stabilization. Our theoretical analysis will focus on the experimental situation in which the limiter is too heavy to be lifted ("*hard* limiter"). In this case superstable orbits are generated, expressing the fact that optimal control has been achieved. Softer limiter control will share the properties derived, with only a reduced degree of stability [8].

Simple limiter control can also be applied to discrete-time systems. Hard-limiter control (HLC) is particularly simple to implement in any dimension, by restricting the phase space with a rigid boundary that cannot be crossed. Figure 1(a) shows the application of this method to the Hénon map $\{x_{n+1}, y_{n+1}\} = \{1 - ax_n^2 + y_n, bx_n\}$, by resetting all values $x < h$ to h. Likewise, for Fig. 1(b), all values $x > h$ were reset to h. Both figures show bifurcation diagrams with differing amounts of stabilized orbits. In addition to stable behavior, we find smeared bifurcation structures, whose occurrences are found to strongly depend on the placement of the limiter. Based on the analysis of the one-dimensional (1D) variant of this problem, we will show that these structures are the result of suboptimal limiters and that they can be removed by a more careful implementation. Simple limiter control in 1D leads to maps whose tops are deformed in order to yield stable periodic orbits. The most efficient control method is obtained for "flat-topped" maps, which corresponds to the HLC case [8]. As their simplest example, we will analyze the flat-topped tent map.

Flat-topped tent maps are obtained by replacing the peak region of a symmetric fully developed tent map by a horizontal straight line at height h, which yields the equation

$$F: x_{i+1} = \begin{cases} w := 1 - |2(x_i - 0.5)|, & \text{for } w \le h, \\ h, & \text{otherwise.} \end{cases} \quad (1)$$

Figure 2(a) shows the map and the bifurcation diagram as a function of the natural control parameter h. It is observed that the controlled map undergoes a period-doubling bifurcation cascade, leading to long, seemingly chaotic orbits. However, in this system, there are no chaotic orbits: Each orbit will eventually pass by the control segment, from where on the orbit is periodic.

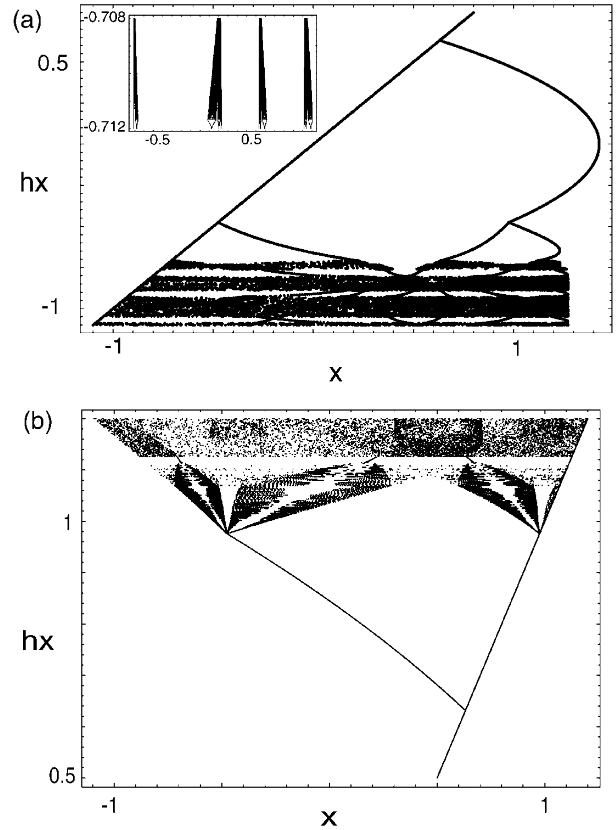

FIG. 1. Straightforward HLC on the Hénon map, showing dependence of bifurcation diagrams on limiter placement. (a) x values smaller than h_x are set to h_x, and (b) x values larger than h_x are set to h_x. The wild structures in the bifurcation diagrams are due to chaotic orbits. Inset: Local emergence of chaos.

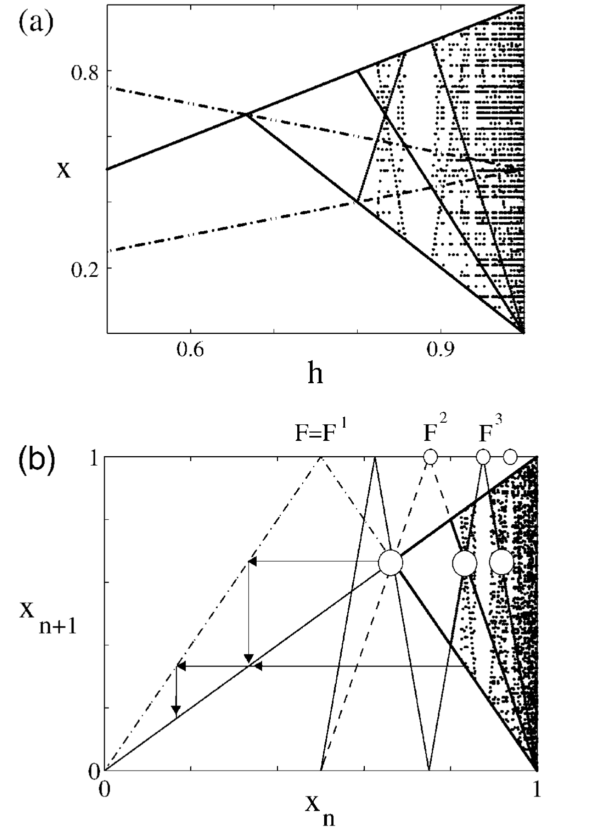

FIG. 2. (a) Bifurcation diagram of the flat-topped tent map. Broken line: inverted map. (b) Relation between the n-fold iterates of F (graphs F^n, $n = 1, 2, 3$ shown by dash-dotted, dashed, and full lines) and the scaling of the "stars" (large circles): Back iterations of $x = 2/3$ (see arrows) yield successive locations of the stars. A similar observation applies for the "windows" (centers indicated by small circles).

The observed period-doubling bifurcation cascade differs in scaling from the Feigenbaum cascade [9]. Because of the constant absolute slope of the map, all branches in the bifurcation diagram are straight lines. Given an orbit of length 2^n, the slope of the bifurcation branch that contains $x = 0.5$ can be written as $s_n = 2^{2^{n-1}}$. The sequence of period-doubling bifurcations can now be calculated as the intersections of this branch with the lines corresponding to the end points of the flat top. For $n > 1$ this leads to

$$h_n = 1 - \frac{\prod_{k=0}^{n-2}(2^{2^k} - 1)}{2^{2^{n-1}} + 1}, \qquad (2)$$

where 2^n, $n > 1$, denotes the periodicity of the cycle, from which $h_\infty \sim 0.824\,908\,067\,280\,21$ is obtained. Note, however, that beyond h_∞, there is no chaos. The convergence towards h_∞ is very fast. In fact, the behavior $\delta^{-1} \sim 2^{-2^n}$ emerges [10]. Constant α can be obtained from the renormalization equation $g(x) = Tg(x) = -\alpha g(g(-\frac{x}{\alpha}))$, yielding $\alpha = 1$. The obtained δ and α apply for all hard-limiter controlled one-dimensional maps. The ratio of the bifurcation fork openings within forks of the same periodicity, however, depends on the derivative of the map, and is therefore nonuniversal. A complete investigation of the scaling properties of hard-limiter control should also contain an explanation of the additional repetitive structures in the bifurcation diagrams [see Fig. 2(b)], which we will call "stars" (indicated by the large circles) and "windows" (the adjacent empty bands). As the locations of the stars are found by back iterating $x = 2/3$, the asymptotic scaling of the stars is given by $F'(0)$. The approximate center of the windows coincides with the outermost maximum of the map F^n (the nth iterate of the map). Subsequent locations can therefore similarly be found by back iterating the neighborhood of point $x = 1$. This shows that the asymptotic scaling of these structures is also determined by $F'(0)$. As a consequence, both scalings are nonuniversal.

With the classical methods, unstable periodic orbits can be controlled only when the system is already in the vicinity of the reference orbit. As the initial transients can become very large, targeting algorithms have been designed to speed up this process [11,12]. HLC makes targeting algorithms unnecessary, as the control-time problem is equivalent to a strange repeller escape (control is achieved, as soon as the orbit lands on the flat top). As a consequence, the convergence onto the selected orbit is exponential. This is corroborated by the escape rate of the map, whose values can be obtained from simulations or via the cycle expansion method [13,14]. As an example of the latter, at $h = 2/3$ the dynamical zeta function is given by $1/\zeta = 1 - z\frac{1}{2}$, and only the cycle at $x = 0$ has to be taken into account. The escape rate $\kappa = \ln(2)$ shows that for arbitrary initial conditions, the probability to land on the period 1 orbit within 5 iterations is $p = 0.95$.

To summarize, 1D HLC systems exclusively exhibit periodic motion, with exponential convergence onto controlled orbits. In the control space, a fast scaling $\delta^{-1}(n) \sim 2^{-2^n}$ emerges. Controlled orbits are true orbits, in terms of the original system, only at bifurcation points of the controlled map. For generic one-parameter families of maps all bifurcation points are regular, and isolated in a compact space. As a consequence, their Lebesgue measure is zero. Questions that emerge in higher dimensions are the following: (1) How should the limiter be implemented, and how should the control be fine-tuned, so as to drive the orbits towards the target orbit? (2) Which of the observed one-dimensional properties will still apply? In Figs. 1(a) and 1(b), the limiter was placed parallel to the y axis, and considerably differing bifurcation diagrams emerged depending on the half-plane that we prohibited. As in the 1D case, controlled orbits are original system orbits only when the action of the controller is marginal, i.e., at bifurcation points of the controlled map, and convergence onto controlled orbits again is exponential. Away from the bifurcations, orbits generally contain points that are distant from the attractor. Their distance is an indicator of the control strength needed to maintain the "artificial" orbit. This can be used experimentally to drive artificial towards true system orbits.

The apparent lack of periodic behavior in Fig. 1(b) indicates that the quality of the control is strongly influenced by the limiter implementation. If we investigate the nature of long orbits in the controlled Hénon systems of Fig. 1, we find that after visiting the limiter, some orbits remain unstabilized, which is in stark contrast to the 1D case. Such orbits are the origin of the smearing of bifurcation diagrams, especially prominent in Fig. 1(b). The reason for this becomes evident if the Hénon system is written as a two-step recurrence equation, e.g., $y_3 = b - \frac{a}{b}y_2^2 + by_1$. Control on the x coordinate alone will not necessarily prevent the existence of chaotic orbits. Consider two generic points $\{x_1, y_2\}, \{x_2, y_2\}$. After an encounter with the controller, their x coordinates will be identical, but their y coordinates will differ. If the limiter is not encountered two successive times, this difference will persist and may lead to a positive Lyapunov exponent. An elementary inspection shows that points with large positive [Fig. 1(b)] and, likewise, large negative x coordinates [Fig. 1(a)] have this property. Note, however, that hitting the controller two successive times is only a sufficient, but not a necessary, condition for control. In the corresponding bifurcation diagrams, chaotic orbits thus overlay the simple structure of stars and windows known from the flat-topped tent map. Chaotic orbits that densely surround bifurcation points are a hindrance for control. To prohibit their generation, it is sufficient to control x and y simultaneously. This can be done in various ways, resulting in bifurcation diagrams that may differ between adjacent bifurcation points. For the generation of Fig. 3(a), we used a controller in the x direction as in Fig. 1(a). After the encounter with the limiter, we controlled the y coordinate by reinjecting the orbit on the lowest foliation of the attractor, at the given x coordinate. Using this technique allows in principle the stabilization on any periodic orbit and removes chaos completely. A close inspection of the recovered bifurcation structure reveals identical scaling properties as encountered in the 1D case.

Continuous-time systems can be treated along the same lines. These systems are even simpler to control than the Hénon map, as their attractors can generally be reduced to almost 1D curves by means of suitable Poincaré sections. We used Hénon's method to compute the standard Poincaré section of the Roessler system. In this section, the application of a straightforward HLC yields identical problems as initially encountered with the Hénon system.

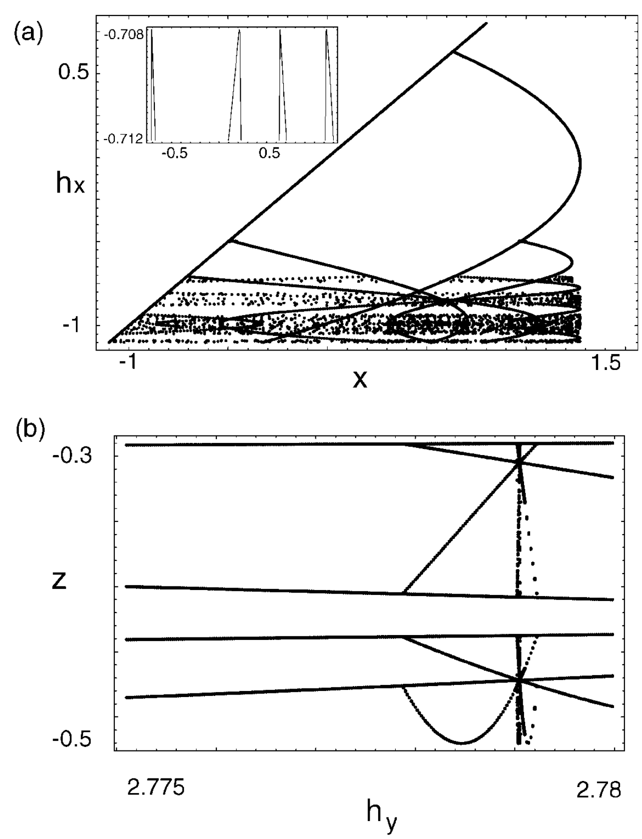

FIG. 3. (a) Bifurcation diagram of the Hénon map for an optimal limiter. Even the smallest areas of chaotic orbits are removed (see inset). The characteristic 1D bifurcations are clearly visible. (b) Details of the bifurcation diagram of the Roessler system [periodicities $p = 4, 8, (16)$], exhibiting the typical 1D bifurcations.

If, however, we interpolate the section by a smooth manifold, onto which we project when control is required, we are able to control all coordinates. Using this approach, HLC works very well and exhibits the properties predicted by the 1D analysis [see Fig. 3(b)].

Despite the fast scaling in the control space, the bifurcation diagram retains rich structures that can be exploited for the selection of orbits with particular properties. It has been shown above that a simple scaling leads from periodic orbits of order 2^n to periodicity of the order 2^{n+1}. More generally, stripes between windows contain whole sets of orbits with particular periodicities (e.g., periodicities of the form $\{3 \times 2^n, 3 \times 2^{n+1}, ..\}$), which greatly simplifies their localization. For the stabilization of orbits of high periodicity, similar observations apply as for the conventional control approaches. In general, orbits become more difficult to stabilize with increasing periodicity. We found it effective to first target an orbit of suitable periodicity, and then to move the limiter carefully towards the optimal control position, until the controlled orbit is sufficiently close to the target orbit. In this way, the stabilization on a period-8 in the Roessler system poses no problems [see Fig. 3(b)]. Once the correct controller placement is known, the time needed to stabilize the orbit is essentially independent of the periodicity. The results of a switching experiment among periodic orbits of the Hénon system are shown in Fig. 4, where a list of correct limiter placements has previously been established. All the experimental procedure further involves is reading off the position corresponding to the desired orbit, and placing the limiter accordingly (for a similar approach, see [15]). The results exhibit a very fast switching time between periodic orbits (extending up to periodicity $p = 16$) that is unaffected by an increased periodicity. Clearly, the extension to hyperchaotic systems is possible along the same lines, if a Poincaré section sufficiently close to 1D can be found.

Compared to the classical approach, where targeting in a high-dimensional space and stabilization is performed consecutively, HLC results in a significant improvement in the time needed to achieve the desired orbit. The combination of targeting and stabilization also is advantageous in the presence of additive noise, as sporadically escaping orbits are automatically recaptured. Although true orbits exist only on a zero Lebesgue measure set of limiter placements, the "correct" locations can be approached with ease. Optimal efficiency in higher dimensions requires the correct implementation of the limiter. The results presented in [8] show that the neighborhood of HLC systems is sufficiently large and general, implying that our theoretical explanation can serve as a general guideline to simple limiter control. Based on our analysis, we conclude that control by simple limiters has the potential to become a powerful tool for the control of unstable orbits.

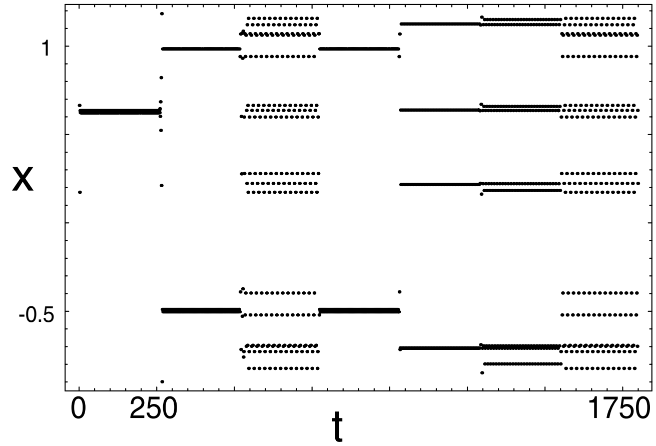

FIG. 4. Fast switching of the Hénon system between periodicities $p = 1, 2, 4, 8, 16$, using HLC. x coordinates are shown over $t = 250$ periods each, a slightly thickened line indicates the position of the limiter. Control of high periodicities is at least as fast as that of low periodicities.

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