PROPERTIES OF THE CONTROL OF NOISY, STABLE AND CHAOTIC DYNAMICS

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Abstract— *When strong multiplicative noise is added to a dynamical system, we find two distinct generic control regimes. In the regime of noisy stable behavior, the application of the control is able to reduce noise, only if the underlying periodicity is correctly taken into account. In the chaotic regime, a shift of the optimal control point away from the noise-free control point is observed, scaling linearly with the noise strength. The strongest control gain emerges for unstable orbits of period-1, which suggests, that natural noisy dynamical systems could preferably be controlled on this orbit.*

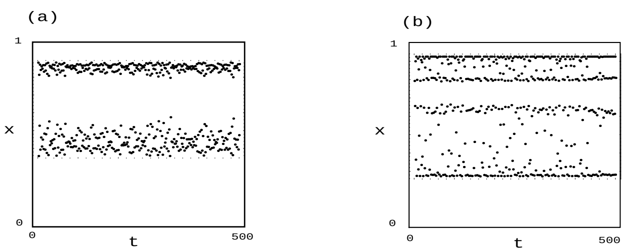

Fig. 1. Noisy time series a) of a superstable period-4 ($str = 0.02$), b) of a chaotic period-4 orbit ($str = 0.01$).

I. Introduction

Recently, ever more applications of dynamical systems approaches deal with biology, living systems and related topics. These systems show amazing self-organization, and efficiency, properties that we hope to learn from, and to make use of for future technology. A particular property of such systems is the presence of noise, on any level of information processing. How is this phenomenon related to the high efficiency properties of natural systems, and how does it affect the controllability of these systems? In this contribution, we shall focus on the second question. We will take the logistic parabola $x_{n+1} = ax_n(1 - x_n)$ as a simple model of a generator of nontrivial dynamical behavior. Onto the logistic parabola, we will load multiplicative noise, that for simplicity we choose uniformly distributed over a finite interval (see Fig. 1). Other distributions (even with diverging cumulants) could be considered and might be more realistic in many applications. However, their explicit form will have no particular impact on the message conveyed by this contribution. The size of the noise sampling interval can be taken as a measure for the noise strength.

The control problem that emerges in this context is as follows: A noisy time series should be controlled so as to remain in a, as small as possible, vicinity of a noise-free system orbit, using as little correction as possible. As a measure dev of the efficiency of the control, for a given noise level, the sum of absolute differences between the noise-free system orbits, and the noisy controlled system orbits, will be taken (involving, for the results reported in this paper, 500 orbit points). To control the noisy system, we will apply the so-called hard limiter control approach (HLC).

II. Control strategies

Elaborate control methods have recently been developed in the context of chaos control. Chaos is composed of an infinite number of unstable periodic orbits of diverging periodicities. In order to exploit this reservoir of characteristic system behavior, methods to stabilize (or ”control”) such orbits using only small control signals have been developed [1], [2], [3], [4]. Practical applications often require that the orbits be quickly targeted and stabilized. For example, the use of unstable orbits for signal transmission in telecommunications would demand a very fast computation of the control signal, as the signal frequency is in the GHz-range. In biology, where control of low-dimensional chaotic firing of neurons [5] is a potential candidate for cortical information encoding, a very efficient control mechanism is required as well. This is implied by a comparison between typical cortical reaction and neuronal inter-firing times ($\sim$ 100 ms vs. $\sim$ 20 ms). A second difficulty for the control in many applications, is the large amount of strong short-time fluctuations. Most control approaches, due to their inherent latency and sophisticated nature, cannot cope with this problem. Recently, Corron et al. [6], [7] introduced a new control approach (termed "control by simple limiters") and suggested, that it could overcome the limitations of the previous methods. The general procedure can be summarized as follows: An external load is added to the system, which limits the phase-space that can be explored. As a result, orbits with points in the forbidden area are eliminated. The authors also observed that the modified systems tend to replace previously chaotic with periodic behavior. When the modified system is tuned in such a way that the control mechanism is marginally effective, the controlled orbit runs in the close neighborhood of an orbit of the uncontrolled system. Recently, the control method was theoretically analyzed and fully explained [8], [9], where an underlying 1-d discrete flat-top map family embodies all properties to be explained.

III. Flat-topped maps

Flat-topped maps are obtained by replacing the peak region of the map by a horizontal straight line at height h representing the limiter in the phase-space. Fig. 2a shows the flat-top tent map with the bifurcation diagram as a function of the natural control parameter h. It is observed that the controlled map undergoes a period-doubling bifurcation cascade, leading to long, seemingly chaotic orbits. However, in this system, there are no chaotic orbits: Each orbit will eventually pass by the control segment, from where on the orbit is periodic. Period-doubling cascades are characterized by two constants, α and δ [10]. The constant α asymptotically describes the scaling of the fork opening by subsequent period doublings, whereas δ represents the scaling of the intervals of period 2^n to those of period 2^{n-1} near the period-doubling accumulation point, i.e. at the transition to chaos. The observed period-doubling bifurcation cascades are typical for flat-top maps (or, of the control method) and differ in scaling from the Feigenbaum case. The ratio of the bifurcation fork openings within forks of the same periodicity now depends on the derivative of the map, and is therefore non-universal. A complete investigation of the scaling properties of HLC should also contain an explanation of the large-scale repetitive structures in the bifurcation diagrams (see Fig. 2b), which we will call "stars" (indicated by the large circles) and "windows" (the adjacent empty bands). It is easy to see that the locations of the stars are found by back-iterating $x = 2/3$. The asymptotic scaling of the stars is therefore given by the derivative of the leftmost fixed-point of the map. The approximate center of the windows coincides with the outermost maximum of the map F^n (the n-th iterate of the map). Subsequent locations can therefore similarly be found by back-iterating the neighborhood of point $x = 1$. This shows that the asymptotic scaling of these structures is also determined by the derivative of the leftmost fixed- point. As a consequence, both scalings are non-universal.

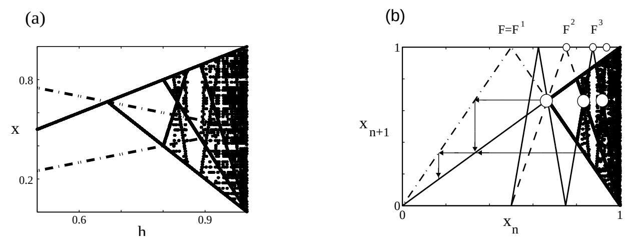

Fig. 2. a) Bifurcation diagram of the flat-topped tent map. Broken line: inverted map. b) Relation between the n-fold iterates of F and the scaling of the "stars" (large circles): Back-iterations of $x = 2/3$ (see arrows) yield successive locations of the stars. Their scaling is therefore determined by the derivative $F'(0)$. A similar argument applies for the size of the "windows" (centers indicated by small circles).

With the classical methods, unstable periodic orbits can only be controlled when the system is already in the vicinity of the target orbit. As the initial transients can become very large, targeting algorithms have been designed to speed up this process [11], [12]. HLC makes targeting algorithms unnecessary, as the control-time problem is equivalent to strange repeller escape (control is achieved, as soon as the orbit lands on the flat top). As a consequence, the convergence onto the selected orbit is exponential. This is corroborated by the escape rate of the map, whose values can be obtained from simulations, or via the cycle expansion method [13], [14]. As an example of the latter, at $h = 2/3$ the dynamical zeta function is given by $1/\zeta = 1 - z\frac{1}{2}$, and only the cycle at $x = 0$ has to be taken into account. We obtain an escape rate of $\kappa = \ln(2)$, implying that for arbitrary initial conditions, the probability to land on the period 1-orbit within 5 iterations is $p = 0.95$. Similarly obtained results coincide with those obtained from simulations. To summarize: 1-d HLC systems exclusively exhibit periodic motion, although period doublings are observed. These period-doubling cascades are not of the Feigenbaum type. In the control space, a fast scaling $\delta^{-1}(n) \sim 2^{-2^n}$ emerges. Controlled orbits are true orbits, in terms of the original system, only at bifurcation points of the controlled map. For generic 1-parameter families of maps all bifurcation points are regular, and isolated in a compact space. As a consequence, their Lebesgue measure is zero.

IV. Numerical results

As HLC can obviously be profitably applied for stable, and for unstable, noisy orbits alike, we may distinguish two control regimes: A) The regime of underlying stable periodic behavior, B) The regime of chaotic dynamics.

A. Regime of stable orbits

For our numerical investigations of the control of noisy stable period-1 and period-2 fixed-points, we chose the superstable orbits (by adjusting to $a = 2$ and $a = 1 + 5^{1/2}$, respectively). For the noise-prone orbits, we calculated the deviation to the noise-free system, as a function of the noise, and of the limiter position. For zero noise, the effect of the control expressed by means of the deviation dev is a piecewise linear function of the limiter position h. The function increases as the limiter is posed ever further below the function maximum, where the coefficient generally is a function of the periodicity and of a. If its position exceeds the function maximum, the limiter has no effect and the slope is zero (see Fig. 3 for period-1).

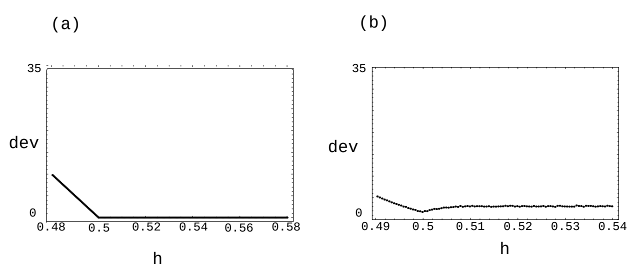

Fig. 3. Dependence of dev on the control point h (summation over 500 orbit points, period-1 orbit). a) A piecewise linear function with a minimum is obtained for zero noise. b) At nonzero noise ($str = 0.02$), the function becomes nonlinear, with a minimum at the optimal noise-free control point).

The control of the noisy superstable period-1 is unproblematic. For nonzero noise, the formerly piecewise linear function becomes nonlinear, with the minimum remaining at the optimal control point of the noise-free system. The optimal control is thus obtained at the optimal control point of the noise-free system. No matter at what noise level, optimal use of the control leads to an improvement, where in the interval of noise $str < 0.1$, which we chose as the interval of realistic noise strength, the deviation is a linear function of the noise strength. The stronger the corrections needed, the lower is h, the more concentrated is the orbit position histogram on the control point. This point, however, generally deviates from the associated orbit point, which leads in spite to an increasing deviation. Controlling on the superstable period-2 yields a similar picture. One difference, however, is that the amount of noise tolerable to maintain control (yielding a significantly reduced value of dev), decreases considerably. Within the controllable region, using the control, we were able to reduce the deviation by a factor of 0.5. For stable orbits, control can beneficially be applied up to relatively large noise levels (see Fig. 3 for $str = 0.08$). Above this level, control is lost. This happens, when due to the noise, the order among the different orbit points is changed.

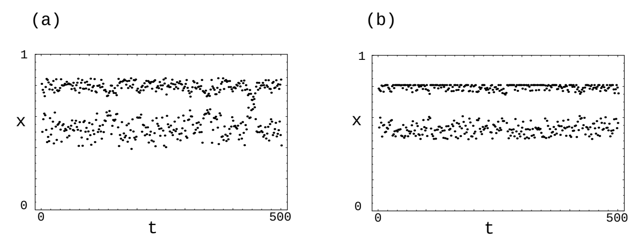

Fig. 4. Time series a) before and b) after control (superstable period-2 orbit, noise level $str = 0.08$).

Under substantial noise, therefore only low-order periodic orbits can be controlled. E.g., for period four, already at a noise level $str > 0.01$, interchange of orbit points sets in. If no importance to the correct ordering of the cycle points is attached, and only the distance individually to the closest cycle points is measured, control beyond the threshold of orbit point interchange may be beneficial. Interestingly, the function $dev(h, str)$ scales linearly with str (identical curves emerge, if h and dev are rescaled by str). Most importantly, however, is that the system is controlled with reference to the natural state of the system. Controlling on non-natural, artificial, periods, leads to disastrous deviations. In this case, much energy is spent on "control friction". Good control results are obtained only when the correct periodicity is controlled at the optimal control point of the noise-free system. By means of the control, as a rule of thumb, the deviation can be reduced by a factor of 0.5.

B. Regime of chaotic orbits

For the chaotic regime, we focus on the fully developed parabola ($a = 4$). Again, to control proper system orbits, the control point must be chosen properly (at the bifurcation points of Fig. 2a). The appropriate location of h can be calculated from the renormalization approach to flat-topped maps [8], or be found experimentally. In the chaotic regime, without applying control, we are unable to remain on a periodic orbit. It is therefore impossible to express the control efficacy as done above. For obtaining the period-1 orbit in the noise-free case, the limiter was adjusted at $h = 0.75$. In the presence of noise, in contrast to the stable regime, the optimal control point moves away from the noise-free control point, where the amount of displacement is essentially a linear function of the noise strength. Again, the deviation dev was proportional to the noise strength. For the unstable period-2 orbit, the noise-free control point is at $h = 0.904$. The displacement of the noisy optimal control point with regard to the noise-free control point is again a linear function of the noise strength. The same observation holds for the minimal deviation. The shift of the control point spans over an interval of more than $\delta h = 0.1$, and is therefore of a size comparable to the added noise (see Fig. 5).

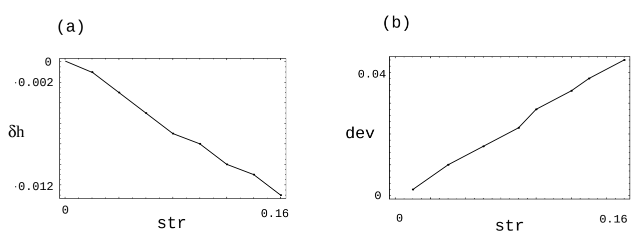

Fig. 5. Controlled unstable period-2 orbit in the chaotic regime, where str has been increased equidistantly. a) Linear dependence of the optimal control point displacement on the noise strength str. b) Linear dependence of dev at the optimal control point, on the noise strength.

Controlling period-4 yields an even stronger shift from the optimal noise-free control point $h = 0.925$. The amount of sustainable noise, however, is again reduced if compared with period-2 (roughly, by a factor of 0.5). Beyond a noise strength of $str = 0.04$, the orbit escapes control.

V. CONCLUSIONS

A correct identification of the underlying system state is crucial. In the stable regime, the results of the control depend very much on the correct identification of the periodicity, where the noisy system can be controlled at the highest noise-free orbit point. In the chaotic regime, noise forces the optimal control point to move away from the noise-free control point, linearly scaling with the noise strength. Controlling at this point, yields a dev-error decrease by roughly one fourth if compared with the one obtained at the noise-free optimal point. Detailed investigations show that the linear shift of the optimal control point, as a function of the noise strength, depends on the nature of the noise. If positive noise is added, the effect vanishes. Although not particularly spectacular, the performance improvements obtained by the application of the control method in both regimes may be significant in the context of biological signal processing.

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REFERENCES

[1] E. Ott, C. Grebogi, and J. A. Yorke, Controlling chaos, Phys. Rev. Lett. **64**, 1196 (1990).

[2] A. Garfinkel, M. Spano, W. Ditto, and J. Weiss, Controlling cardiac chaos, Science **257**, 1230 (1992).

[3] H. Schuster, *Handbook of Chaos Control* (Wiley - VCH, Weinheim, 1999).

[4] I. Marino, E. Rosa, and C. Grebogi, Exploiting the natural redundancy of chaotic signals in communication systems, Phys. Rev. Lett. **85**, 2629 (2000).

[5] R. Stoop, D. Blank, A. Kern, J.-J. v.d. Vyver, M. Christen, St. Lecchini, and C. Wagner, Collective bursting in layer IV - Synchronization by small thalamic inputs and recurrent connections Cog. Brain Res. **13**, 293 (2002).

[6] K. Myneni, T. Barr, N. Corron, and S. Pethel, New method for the control of fast chaotic oscillations, Phys. Rev. Lett. **83**, 2175 (1999).

[7] N. Corron, S. Pethel, and B. Hopper, Controlling chaos with simple limiters, Phys. Rev. Lett. **84**, 3835 (2000).

[8] C. Wagner and R. Stoop, Renormalization approach to optimal limiter control of 1d chaotic systems, J. Stat. Phys. **55**, 97-107 (2002).

[9] R. Stoop and C. Wagner, Scaling properties of simple limiter control, Phys. Rev. Lett., in press (2003).

[10] M. Feigenbaum, The universal metric properties of nonlinear transformations, J. Stat. Phys. **21**, 669 (1979).

[11] T. Shinbrot, W. Ditto, C. Grebogi, E. Ott, M. Spano, and J. Yorke, Using the sensitive dependence of chaos to direct trajectories to targets in experimental systems, Phys. Rev. Lett. **68**, 2863 (1992).

[12] E. Kostelich, C. Grebogi, E. Ott, and J. Yorke, Higher dimensional targeting, Phys. Rev. E **47**, 305 (1993).

[13] R. Artuso, E. Aurell, and P. Cvitanovic, Recycling of strange sets: I. Cycle expansions, Nonlinearity **3**, 325 (1990).

[14] R. Artuso, E. Aurell, and P. Cvitanovic, Recycling of strange sets: II. Applications, Nonlinearity **3**, 361 (1990).