Global Dynamics of Finite Cellular Automata

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Abstract. A novel algebraic and dynamic systems approach to finite elementary cellular automata is presented. In particular, simple algebraic expressions for the local rules of elementary cellular automata are deduced and the cellular automata configurations are represented via Fourier analysis. This allows for a further analysis of the global dynamics of cellular automata by the use of tools derived from functional analysis and dynamical system theory.

1 Introduction

Cellular automata (CA) models have been widely studied and applied in physics, biology and computer science. They are amongst the simplest systems which exhibit self-organisation and are closely related to neural networks. CA have also been suggested as the generic discrete model of pattern formation and decentralised computation. Despite their simple appearance there are a number of important open theoretical questions [1,2,3,4].

Contrary to the usual mathematical treatment we consider here *finite* cellular automata. This starting point ensures a close analogy to actual biological systems and may eventually be generalised in the infinite size limit.
A cellular automaton is specified by a d-dimensional regular discrete lattice L with boundary conditions and a finite set Σ of states x_i assigned to each node or cell i of the lattice. A *local rule* f acting on the cells in the range k of the *neighbourhood* N_k^i of each cell i determines the dynamics of the cellular automaton in discrete time steps starting from an initial condition. In this study we consider only *finite* CA, that is CA with a finite number N of cells.

Unless stated otherwise, only *elementary* cellular automata will be considered, that is cellular automata with $d = 1$, $\Sigma = \{0, 1\}$ and nearest neighbourhood $k = 3$. In this case, there are 256 different possible local rules ${x_i}^{t+1} = f({x_{i-1}}^t, {x_i}^t, {x_{i+1}}^t)$. The N cells are subject to *periodic boundary conditions* and their states x_i are updated synchronously by the local rule. Local rules are given by a *rule table.*

Example 1 (rule 110). The rule table of CA rule 110 is $(f(111) = 0, f(110) = 1, f(101) = 1, f(100) = 0, f(011) = 1, f(010) = 1, f(001) = 1, f(000) = 0)$.

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It is customary to assign a decimal number to such rule tables. One speaks of *rule 110* as the binary expansion of the decimal number 110, that is 110 = 01101110, encodes the rule table when read from left to right.

A configuration $x^t = (x^t_0, x^t_1, ..., x^t_{N-1})$ of a CA is the string of the states of the N cells at the time t. Starting from an initial configuration x^0, the global function or map F then maps configurations x^t to subsequent configurations x^{t+1} thereby generating a space-time pattern. The global map F is only indirectly given through the local rule f. Figure 1 shows examples of space-time patterns generated by the CA rules 90 and 110.

Understanding the global dynamics induced by the global map F is the long-standing challenge of CA theory. CA with a finite number N of cells will eventually always become periodic (after at most 2^N steps). However, already the rather small CA of figure 1 with 65 cells could yield an intractable $2^{65} \sim 3.7 \cdot 10^{19}$ time steps. In this paper we present a novel approach in order to further analyse the global dynamics of finite CA. The outline of the paper is as follows: First, we introduce algebraic expressions for the local CA rules. Second, a continuous representation of the CA configuration space by Fourier analysis is presented. We then discuss the implications and possibilities of this approach for finite CA in view of functional analysis and dynamic system theory.

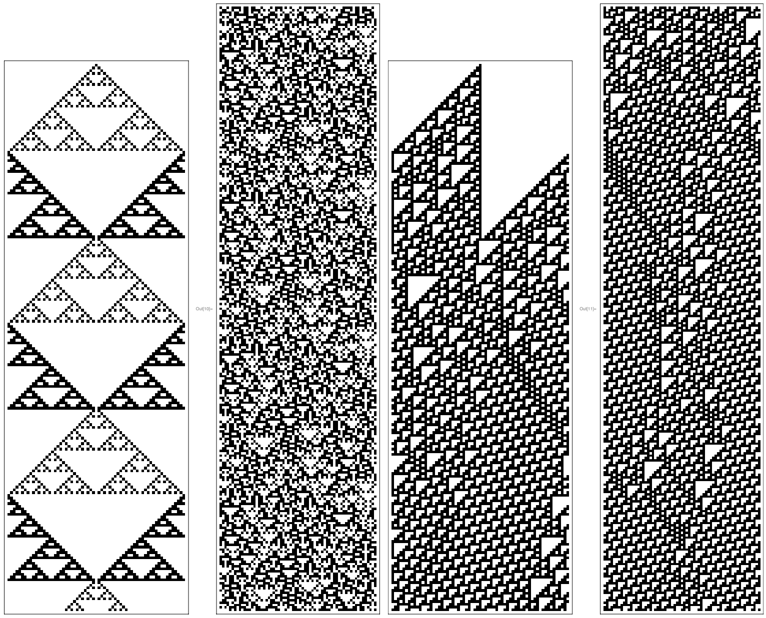
2 Algebraic Expressions of Local Rules

If a CA rule table is viewed as a truth table from propositional logic it is immediately clear that every rule table represents a *Boolean function*, which must be expressible as a *disjunctive normal form* (DNF) [5]. The DNF is a disjunction of clauses, where a clause is a conjunction of Boolean variables.

Example 2 (DNF of rule 110). CA rule 110 written as a DNF is $(X_{i-1} \wedge X_i \wedge \neg X_{i+1}) \vee (X_{i-1} \wedge \neg X_i \wedge X_{i+1}) \vee (\neg X_{i-1} \wedge X_i \wedge X_{i+1}) \vee (\neg X_{i-1} \wedge X_i \wedge \neg X_{i+1}) \vee (\neg X_{i-1} \wedge \neg X_i \wedge X_{i+1})$ with the Boolean variables X_i, the disjunction denoted by $\vee$, the conjunction by $\wedge$ and the negation by $\neg$.

The representation of cellular automata rules as Boolean functions has in fact been used by Wolfram to program cellular automata rules [6]. However, this line of research seems to have not yielded any further insights into CA rule dynamics [2]. We propose to translate the DNF of CA rules to algebraic expressions which will allow for a further analysis of CA dynamics. A similar idea but within a different framework was also proposed by Chua [8]. In the representation chosen here, the conjunction $X_i \wedge X_j$ is expressed by the algebraic multiplication, $x_i \cdot x_j$, the disjunction $X_i \vee X_j$ is expressed by the algebraic relation $x_i + x_j - x_i \cdot x_j$ and the negation $\neg X_i$ is expressed by $1 - x_i$. All CA rules ${x_i}^{t+1} = f({x_{i-1}}^t, {x_i}^t, {x_{i+1}}^t)$ expressible as Boolean functions can be written in such a form, where the state of the cell i at time $t+1$ is given by an algebraic expression of the neighbourhood states at time t.

Example 3 (Algebraic expressions of rule 110 and rule 90). The algebraic approach proposed here is discussed by example of the prototypic rules 110 and



a b c d

Fig. 1. Space-time patterns generated by CA rules 90 and 110. The vertical axis is the time axis. **a** Space-time pattern of rule 90 applied on a initial configuration of 65 cells of which all are in state 0 except one being in state 1 for 250 iterations. **b** Space-time pattern of rule 90 applied on a random initial configuration of 65 cells for 250 iterations. **c** Space-time pattern of rule 110 applied on a initial configuration of 65 cells of which all are in state 0 except one being in state 1 for 250 iterations. **d** Space-time pattern of rule 90 applied on a random initial configuration of 65 cells for 250 iterations.

90. Rule 110 is amongst the most complex elementary CA rules as it has been proven that it is capable of universal computation in the Turing sense [14]. Rule 90 is among the simplest elementary CA rules as one of the so-called additive CA rules [7].

The Boolean formula (DNF) for CA rule 110 becomes

$$ {x_i}^{t+1} = {x_i}^t + {x_{i+1}}^t - {x_i}^t \cdot {x_{i+1}}^t - {x_{i-1}}^t \cdot {x_i}^t \cdot {x_{i+1}}^t \quad (1) $$

and for CA rule 90 (=01011010)

$$ {x_i}^{t+1} = {x_{i-1}}^t + {x_{i+1}}^t - 2{x_{i-1}}^t \cdot {x_{i+1}}^t. \quad (2) $$

Elementary CA rules are grouped according to the 0-1-transformation and the *left-right*-transformation of the rules yielding 88 independent groups of rules which essentially display the same global dynamics [9]. Within the algebraic approach proposed here these transformations are simple algebraic operations on the CA rules. The $0-1$-transformation T_{0-1} is given by $T_{0-1}f(x_{i-1}, x_i, x_{i+1}) = 1 - f(1-x_{i-1}, 1-x_i, 1-x_{i+1})$ and the left-right transformation $T_{left-right}$ is given by $T_{left-right}f(x_{i-1}, x_i, x_{i+1}) = f(x_{i+1}, x_i, x_{i-1})$.

Example 4 (The group of rules equivalent to rule 110). The group of rules equivalent to rule 110 under the 0-1-transformation and the left-right-transformation: Rule 110 (=01101110)

$$x_i{}^{t+1} = x_i{}^t + x_{i+1}{}^t - x_i{}^t x_{i+1}{}^t - x_{i-1}{}^t x_i{}^t x_{i+1}{}^t \tag{3}$$

Rule 137 (=10001001) (0-1-transformation)

$$x_i{}^{t+1} = 1 - x_{i-1}{}^t - x_i{}^t - x_{i+1}{}^t + x_{i-1}{}^t x_i{}^t + x_{i-1}{}^t x_{i+1}{}^t + 2x_i{}^t x_{i+1}{}^t - x_{i-1}{}^t x_i{}^t x_{i+1}{}^t \tag{4}$$

Rule 124 (=01111100) (left-right-transformation)

$$x_i{}^{t+1} = x_{i-1}{}^t + x_i{}^t - x_{i-1}{}^t x_i{}^t - x_{i-1}{}^t x_i{}^t x_{i+1}{}^t \tag{5}$$

Rule 193 (=11000001) (0-1-transformation and left-right-transformation)

$$x_i{}^{t+1} = 1 - x_{i-1}{}^t - x_i{}^t - x_{i+1}{}^t + 2x_{i-1}{}^t x_i{}^t + x_{i-1}{}^t x_{i+1}{}^t + x_i{}^t x_{i+1}{}^t - x_{i-1}{}^t x_i{}^t x_{i+1}{}^t \tag{6}$$

Note that all CA rules with $\Sigma = \{0, 1\}$ can be assigned by this method to an algebraic expression. In fact, this approach allows to derive an algebraic expression for any kind of network which operates with Boolean functions.

If the variables x_i are drawn from a probability distribution we get algebraic expressions for probabilistic elementary CA which yield in the mean field approach the familiar mean field equations of probabilistic elementary CA [4,6]. To the best of our knowledge, we have here for the first time simple algebraic expressions for *all* elementary CA rules. Previous studies either examine only restricted classes of elementary CA rules, mostly *additive* CA rules [7] or cannot yield the same simple forms as our approach [8]. Of course, the algebraic expressions derived here apply equally to infinite elementary cellular automata.

In general, the algebraic approach proposed here will, through the equations of the form (1), give a *system of coupled nonlinear Boolean difference equations* for the time evolution of the global configurations x^t. We will not pursue this line further in this paper but instead investigate a continuous representation of CA configurations in order to analyse global CA behaviour.

3 Global Functions and Dynamics of Finite Cellular Automata

The global dynamics of a cellular automaton is determined by repeatedly applying the global map F to some initial configuration x^0. The global map F is only indirectly given by the local map f. One might be tempted to assign rational or real (for infinite configurations) numbers to the configurations x^t [6,10].

This approach has the drawback that it induces a wrong topology in the CA configuration space. The set of all possible CA configurations of length N forms a *N-dimensional hypercube* with the *Hamming distance* as the natural metric. The Hamming distance counts the number of bits for which two binary strings differ and is here defined by $d_H(x,y) = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-y[n])^2$ for two CA configurations, i.e. two binary strings $x[n]$ and $y[n]$ of length N. We are looking for a transformation which preserves the topology induced by the Hamming metric. In the usual dynamic system approach to CA, infinite size CA in d dimensions are treated as continuous functions on the compact metric space Σ^{Z^d} equipped with the product topology [1]. This topology for CA as dynamical system has been criticized in [11]. Here, we suggest a different approach for the topological dynamics of finite CA.

We propose to treat the CA configurations $x = x[n], n = 0, ..., N-1$ as a sample of a continuous periodic signal $x(s)$. A continuous signal with period 1 has a Fourier series

$$x(s) = \sum_{k=-\infty}^{\infty} \hat{x}_c(k)e^{2\pi iks} \tag{7}$$

with $\hat{x}_c(k)$ being the Fourier coefficients. If the signal is band limited and sampling is at a rate higher than the Nyquist rate the Fourier coefficients will up to a constant factor be equal to the *discrete* Fourier transform values $\hat{x}(k)$[12]. That is $\hat{x}(k) = N\hat{x}_c(k)$ with the discrete Fourier transform

$$\hat{x}(k) = \sum_{n=0}^{N-1} x[n]e^{-\frac{2\pi ikn}{N}}, 0 \le k \le N-1. \tag{8}$$

We thus have in (7) only a finite summation yielding a continuous function $x(s)$ which returns the initial configuration $x[n]$ when sampled at the rate $\frac{1}{N}$.

The global dynamics of CA crucially depend on the initial configurations. Our approach takes this fact into account as simple or periodic initial configuration will automatically yield simple forms through the discrete Fourier transform (8).

The functions $x(s)$ of equation (7) with finite N form a finite Hilbert space $L_2[0,1]$ with the inner product $(x,y) = \int_0^1 x(s)^* y(s)ds$, where $x(s)^*$ denotes the complex conjugate of $x(s)$. The distance metric is given by the inner product $\int_0^1 (x(s)-y(s))^*(x(s)-y(s))ds$ which results in

$$\begin{aligned} d(x,y) &= \int_0^1 (x(s)-y(s))^*(x(s)-y(s))ds && (9)\\ &= \frac{1}{N^2}\sum_{k=0}^{N-1}(\hat{x}(k)-\hat{y}(k))^*(\hat{x}(k)-\hat{y}(k)) && (10)\\ &= \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-y[n])^2 && (11)\\ &= d_H(x,y) \end{aligned}$$

We thus have through the discrete Fourier transform a distance-preserving isomorphism between two metric spaces, that is between a finite Hamming space (hypercube) and a finite Hilbert space $L_2[0,1]$. The distance measure in Fourier space is given by (10). All dynamical properties of CA (e.g. fixed-points) can accordingly be studied in Fourier space. In general, the dynamics will be given by a nonlinear map in the Fourier space which will be the focus of future work. Additive CA rules [6] however result in simple closed forms irrespective of the initial conditions.

Example 5 (Global dynamics of rule 90). For the additive rule 90, the algebraic expression of the local rule in equation (2) can be shortened to ${x_i}^{t+1} = ({x_{i-1}}^t + {x_{i+1}}^t)\ mod\ 2$. Inserting ${x_i}^{t+1}$ into the Fourier series expression (7) yields

$$x(s)^T = \frac{1}{N}\sum_{k=0}^{N-1}(2\cos(\frac{2\pi k}{N}))^T \hat{x}(k)e^{2\pi iks} mod 2 \tag{12}$$

for the temporal evolution of the CA with T being an arbitrary time step. Note that this equation applies to any initial configuration and is therefore more general and simpler than previous results [7]. It might be surprising that the rather complicated looking dynamics of figure 1 (b) is generated by such a simple form.

Our approach yields through (7) the well behaved functions of a finite Hilbert space $L_2[0,1]$. Accordingly, results from functional analysis can readily be applied. As a first example consider the *Banach fixed-point theorem.*

Theorem (Banach fixed-point theorem for finite CA). *If the global function $F : X \to X$ is k-contractive:*

$$d(Fx, Fy) \leq kd(x,y), \forall x, y \in X \tag{13}$$

and fixed $k, 0 \leq k \leq 1$, the global function F has exactly one fixed point $x*$ on the closed set X. That is there is a unique fixed point, a single homogeneous state the cellular automaton evolves to.

Proof. The classic Banach fixed-point theorem from functional analysis holds because all necessary conditions are satisfied within our approach [13]. □

Only the trivial rules 0 and 255 are contractive and therefore yield single homogeneous states. In ongoing work we study subsets of configurations which yield unique fixed points for certain rules through a (in this regard) restricted Banach fixed-point theorem. The crucial point is proving that these subsets are closed. As the functions $x(s)$ live in a finite Hilbert space, which is in a certain regard the most convenient space for analysis, we expect that more results from functional analysis will turn out to be useful in the analysis of global CA dynamics.

4 Concluding Remarks

Despite the simplicity of the local update rules for elementary cellular automata, the prediction of the global dynamics is a difficult problem, in its generality unsolved to date. In fact, in the infinite case, it has been proven that rule 110 is capable of universal computation and its global dynamics are therefore, in general, intractable [14].

Here, the idea is to tackle the problem by associating the configurations of a finite cellular automaton with states of a continuous state space, allowing for an analysis from a viewpoint of dynamic system theory. Previous approaches following this idea suffered from the fact that, first, there were no simple algebraic expressions for the local CA rules and, second, the association map did not preserve the topology of the original configuration space, the N-dimensional hypercube induced by the Hamming distance. This led to irreconcilable problems in the analysis of the global dynamics. In this contribution we showed:

1. that the rules of a elementary cellular automaton can be translated into algebraic expressions, yielding the possibility of an interpretation as continuous maps,
2. that on the basis of Fourier series, a continuous representation of the configurations can be introduced, which preserves the topology of the configuration space. This leads to a description of the global dynamics of a elementary cellular automaton in terms of the dynamics in a finite Hilbert space $L_2[0,1]$.

This preliminary contribution focused on illustrating the method characterised by these two main points. We believe that this approach opens up various new possibilities in the analysis of global dynamics of CA. We believe that the problems emerging within our approach can be amenable to a rigorous mathematical treatment, this is however outside of the scope of this contribution. Ongoing and future work will focus on studying non-additive CA from a dynamic system perspective. In addition, the Fourier series representation in the Hilbert space $L_2[0,1]$ stipulates further analysis of the dynamics from a functional analysis perspective. Finally, our approach shall in some weaker sense be generalised in the thermodynamic limit, that is the infinite size limit $N \to \infty$.

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