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Shiner–Davison–Landsberg complexity revisited

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Abstract. Shiner, Davison and Landsberg have recently proposed a measure of complexity that has become the subject of an intense debate. We show that using the framework of the thermodynamic formalism, the properties and shortcomings of this measure—over-universality and a trivial implementation of the temperature dependence—can be interpreted and elucidated in a coherent way. Moreover, we show how the SDL approach can be refined to nullify these critiques. Results of the logistic parabola family demonstrate the improved behaviour of the modified SDL measure of complexity. For the tent map family, an interesting linear dependence of the modified measure as a function of the asymmetry is observed.

Keywords: classical phase transitions (theory), exact results, fluctuations (theory), new applications of statistical mechanics

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1. Introduction

A simple and natural approach to measuring complexity put forward by Shiner *et al* [1] has recently fuelled a widespread debate on the subject in general, and on their new complexity measure in particular (henceforth abbreviated as the SDL measure).

The notion of complexity in dynamical systems is common and widely used. However, many different approaches are used to measure this quantity; references [1]–[22] cover but a subset of direct contextual relevance. This expresses the many different facets that complexity has, and raises the question of whether frameworks of complexity could be found within which at least some of the more important extant notions could be systematically embedded. In a recent paper [2], we put forward such an approach that allows a free choice of the observable whose complexity of predictability is to be assessed. In simple terms, the complexity of predictability is defined as the averaged difficulty of inferring future observations from past ones. In this contribution, we use the underlying framework [2, 12] to clarify the properties of the SDL measure. We show that the main shortcoming of the SDL measure is in its trivial implementation of the temperature dependence. In a first extension of the results obtained in [1], we evaluate the SDL measure for the tent and the logistic map families for the case of a dynamical partition of the support. This allows a direct estimation of how much a freely chosen partition influences the results, and how much the SDL concept differs from our definition of the complexity of prediction [2]. We then propose a refinement of the SDL approach, in order to nullify the main point of criticism. Results of this modification are presented for the tent map and the logistic parabola families. They demonstrate that the refinement has a strong effect on the measure. For the tent map family, an interesting linearity property of the measure depending on the family parameter emerges.

Over the last decade, the quest for 'natural' complexity measures has continually attracted interest, from both theoretical and experimental points of view [1]–[20]. Among the different concepts proposed for the characterization of complexity, the Kolmogorov/Solomonoff algorithmic complexity [4, 5, 18] has possibly been the most influential concept. The algorithmic complexity A of an object s is defined as the length of the shortest program P (in bits) that produces (prints) the object s:

$$A(s) := \min_{C,P:C(P)=s} \log(\text{length}(P)), \tag{1}$$

where C is a computer. As there exists a universal computer, called the Turing machine, which is able to simulate any other computer, $A(s)$ is a well-defined quantity. In this way, the algorithmic complexity has been devised as a measure of the complexity of objects generated by computers, or computer programs.

By assigning maximal complexity to random sequences, this measure, however, violates a very basic concept of complexity. From the point of view of an intrinsic notion of complexity, truly random sequences appear no more complex than pseudo-random sequences, even though the latter have a much shorter description length. In fact, computer-generated random sequences are the result of simple random generators, that have finite algorithmic complexities. In the physical world, the distinction between random and pseudo-random sequences is largely irrelevant. As a consequence of Gödel's theorem [23], the values of small digits in measurements are unpredictable *per se*, due to the coupling to the rest of the world. The description of any biological or physical system is thus naturally based upon cylinders of real numbers, even though the measurements may be given in terms of rational numbers. Sequences contained in the same cylinder, however, will have divergent shortest descriptions. This, in few words, is the deeper reason why for natural and physical systems, the algorithmic complexity fails to provide a suitable measure of complexity.

2. The SDL complexity measure and its relation to the fluctuation spectrum

For a measure of complexity one should require that it be zero for truly random, as well as completely ordered, objects. To be useful, it should be easy to evaluate; in particular, it should not require a hierarchical decomposition of the system. Starting from this position, Shiner, Davison and Landsberg [1] defined their complexity measure as

$$\Gamma_{\alpha,\beta} := \Delta^{\alpha}(1-\Delta)^{\beta}, \qquad \alpha, \beta \in \mathbb{R}, \tag{2}$$

where the 'disorder' Δ was defined as $\Delta := S/S_{\max}$, with S being the Boltzmann–Gibbs–Shannon entropy and $S_{\max}$ the maximal entropy. Consistently, $(1-\Delta)$ was defined as the order in the system. The rescaling by $S_{\max}$ maps measured order or disorder into the unit interval. Upon variation of the disorder strength α and the order strength β, the authors hoped to be able to smoothly connect all relevant classes of complexity.

As has been pointed out by several authors [21, 22], this measure of complexity, however, has some important shortcomings. Elucidating the origin of these and pointing out ways to correct them constitute the main content of our contribution. In order to achieve this goal, we reinterpret SDL's work in terms of the fluctuation spectrum [2]. For an observer-chosen variable ε, the associated fluctuation entropy spectrum $S(\varepsilon)$ is derived, using the thermodynamic formalism of dynamical systems [12]. This is achieved by a Legendre transformation, applied to the free energy associated with the natural partition sum induced by the temporal evolution of the system. Formally, the thermodynamic formalism departs from a partition function $Z(n, \beta, \varepsilon)$, where n is the level or depth of the partition and β can be viewed as an inverse temperature. With $Z(n, \beta, \varepsilon)$, a free energy

$$F(\beta) := \lim_{n\to\infty} \left(\frac{1}{n}\right) \log(Z(n, \beta, \varepsilon)) \tag{3}$$

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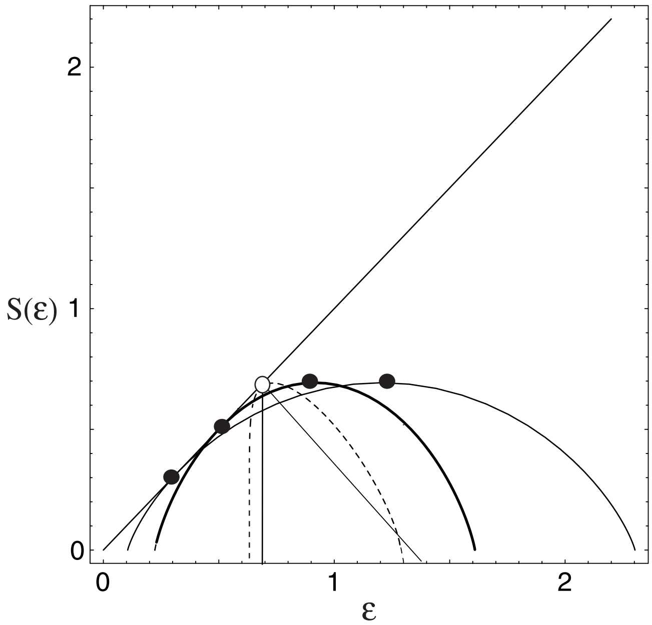

Figure 1. Fluctuation spectra of different maps and specific entropy measures S_I and $S_{\max}$. Solid lines, filled dots: convex entropy functions $S(\varepsilon)$ obtained for two asymmetric tent maps of varying asymmetry. Dashed line, open dot: numerical approximation of $S(\varepsilon)$ obtained for the fully developed parabola (partition level $n = 12$), which slowly converges towards the triangular function (thin solid lines, open dot). In this case, S_I and $S_{\max}$ coincide. In the presence of first-order phase transitions, piecewise linear parts of the graph emerge, as is demonstrated by the parabola.

is associated, where in $F(\beta)$ we suppressed the dependence on the observable. β can be interpreted as an artificial temperature (that has no absolute zero, though). In the absence of phase transitions, a large deviation (or fluctuation) entropy is obtained by means of the Legendre transform

$$S(\varepsilon) := \varepsilon\beta - F(\beta). \tag{4}$$

The requirement that applies to entropy functions is strict convexity with infinite derivatives at the two end-points of the curve (in the absence of phase transition effects). The fluctuation spectrum generally has the convex form shown in figure 1. There, entropy functions for different one-dimensional maps are displayed, for the logarithmic length scale $\varepsilon(x) := \log(|f'(x)|)$ [2] as the chosen observable. In the presence of first-order phase transitions, straight-line parts emerge, as is shown by the example of the fully developed parabola (the thin line in figure 1).

In the context of the fluctuation entropy $S(\varepsilon)$, $\Gamma_{\alpha,\beta}$ has a simple interpretation: S in the SDL formula corresponds to the observable measure S_I in the fluctuation spectrum, defined by $S_I(\varepsilon) = \varepsilon$, whereas $S_{\max}$ corresponds to the topological entropy (the maximum of $S(\varepsilon)$). The basic ingredient of SDL's work is therefore proportional to the product $S_I(S_{\max} - S_I)$. Geometrically, this quantity amounts to the grey area of figure 2.

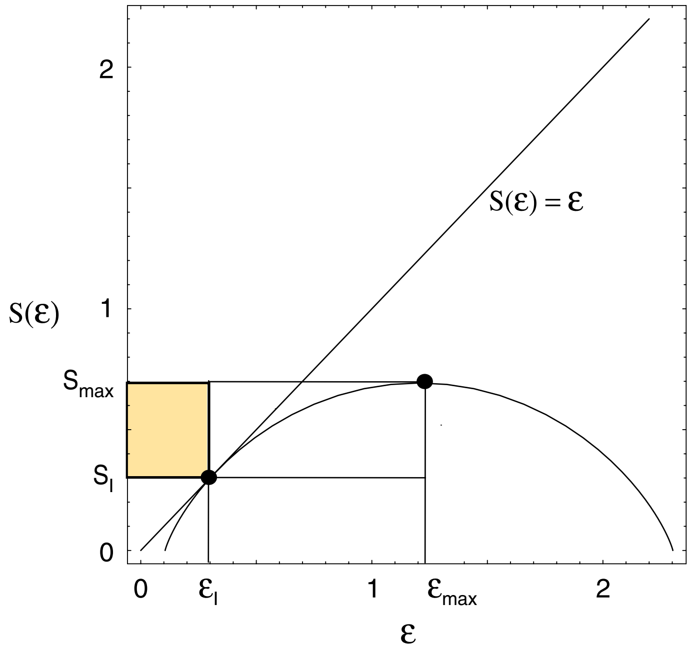

Figure 2. Geometric meaning of the SDL measure: the shaded area has the size $S_I(S_{\max} - S_I)$. For the fully developed parabola (see figure 1), zero area would be obtained.

3. Properties of the SDL measure

By construction, the SDL measure only depends on two particularly significant points of the fluctuation spectrum: the natural measure S_I and the topological measure $S_{\max}$ (the latter is sometimes also referred to as the balanced measure). As a consequence, the remaining shape of the fluctuation spectrum $S(\varepsilon)$ does not influence the measure. This leads to the undesirable situation that very different dynamical systems are characterized by identical complexities $\Gamma_{\alpha,\beta}$, for all values of the exponents α, β. This is illustrated by hyperbolic maps versus maps displaying phase transition phenomena. To most hyperbolic maps, intermittent maps can be found that have identical values of S_I and of $S_{\max}$, and hence identical SDL complexity. Obviously, however, the intermittent maps are much more difficult to predict than hyperbolic ones. An extreme case is obtained for $S_I = S_{\max}$. This property is characteristic for maps with vanishing fluctuation spectrum (e.g., symmetric tent maps); however, some intermittent maps also satisfy this condition.

As a minor point, SDL's work did not specify on what partition their approach should be based. In the main application of their measure, they used a uniform partition. In principle, the choice of a partition corresponds to a selection of the observable and could, therefore, be considered a matter of personal choice. However, as a measure of complexity should be tied to a hierarchical process (in our case to the refinement of the resolution in space or in time), some choices are better suited than others, in particular from the viewpoint of convergence. For the definition of the topological entropy, the ϵ-spanning minimality requirement is in most cases equivalent to saying that $S_{\max} = \log(\text{number of monotone branches of } f)$. With this view, using a binning

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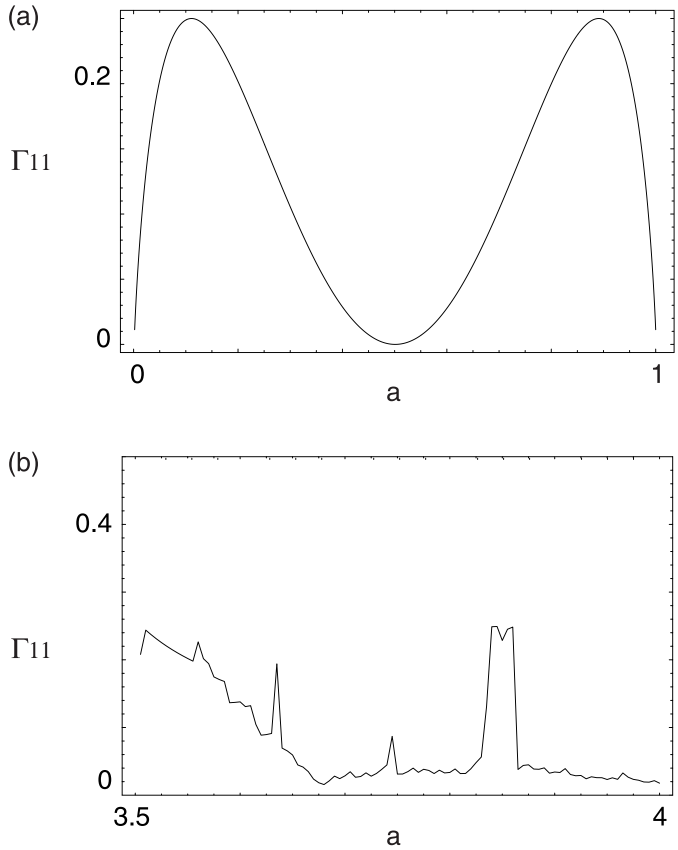

Figure 3. SDL complexity $\Gamma_{1,1}$ based on dynamical partitions for (a) the tent map family, as a function of the asymmetry a, and (b) the logistic map, as a function of the order parameter a.

into $2^{10} = 1024$ bins is in principle not bad. For the 'natural' measure S_I, however, the choice of the binning procedure is of more importance. To provide an illustration, take a uniform binning and consider that half minus one points of a period 2^{11} fall into one bin each, whereas the rest go into the remaining bin. The entropy obtained for this setting then vastly differs from the asymptotic one. Using natural dynamic partitions for the tent map and the parabola family, we obtain SDL complexities as shown in figure 3. These results differ substantially from those obtained using the uniform partition and from those obtained with our $C_{\mathrm{s}}(1,1)$ measure [2], although common trends are observed. To verify this, compare figure 3(b) with figure 3 of [1], and figure 3(a) with figure 2(a) of [2].

4. Modification of the SDL measure

To obtain improved measures based on the order–disorder approach, at least the trivial temperature dependence of the SDL measure should be replaced by a dependence that is better able to reflect differences among systems. This is achieved by the following construction. Recall that the basic SDL measure is proportional to the product $S_I(S_{\max} - S_I)$. Instead of using for the characterization of the system two entropy points

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only, we propose to perform an integration over the entropy contributions from all length scales ε. For each invariant, though, with the exception of the natural measure defined by $S(\varepsilon) = \varepsilon$, the unobservable measure, the length scale ε assumes the role of its maximal entropy, and $S(\varepsilon)$ the role of its natural entropy. The appropriate integrand then has the form

$$(1 - S(\varepsilon)/\varepsilon)^{\alpha}(S(\varepsilon)/\varepsilon)^{\beta}, \tag{5}$$

which can be calculated from the large deviation entropy $S(\varepsilon)$. The results of this calculation for the tent map and the logistic map family are shown in figure 4 for $\alpha = \beta = 1$. We see that the SDL measure is closely related to the complexity measures based on the integrand

$$(S(\varepsilon)/\varepsilon)^{\gamma}, \tag{6}$$

for which a simple interpretation in terms of a complexity of predictability has been given [2]. Unfortunately, for the modified integrand, a corresponding straightforward interpretation seems hardly possible, since in the SDL approach, the weights of intermittent length scales, which characterize the difficulty of prediction [2], are counterbalanced by the first factor of (5). For $\gamma = 1$, the integrand (6) can be interpreted as the fractal dimension of the set of (generically) unobservable measures characterized by the scaling index ε. It is, however, not obvious that this fact could be used for a straightforward interpretation of the SDL complexity.

5. Results and conclusions

Our modification removes the insensitivity of SDL's work to system differences. A comparison with the original SDL complexity reveals that the modification generates notable differences even for fixed weighting exponents. To see this, compare the results obtained for the tent map family (figures 4(a) and 3(a)). Remarkable in the result obtained for the tent map family is that an almost linear dependence of the modified complexity on the asymmetry is obtained, with a very strong decay close to the completely asymmetric situation. Whereas SDL's work displays an overall monotonically decreasing function for the logistic parabola family, this no longer holds for the modified version (compare figure 4(b) with figure 3(b), and with figure 3 of [1]). The original as well as our results based on the dynamic partition exhibit high complexities for completely ordered systems (period-doubling cascade cases or period-3 windows). As an expression of complexity, this appears difficult to accept, as it misses one condition for measures of complexity that we would like to see fulfilled: that the complexity should be zero when the system is either completely ordered or random. For our modified measure, this shortcoming is removed. As Crutchfield *et al* [21] pointed out, any measure of complexity must be tied intrinsically to a process. By putting the measure on the basis of the thermodynamic formalism, a coherent interpretation of the SDL measure could be given, and the importance of the chosen partition was demonstrated. Starting from the ansatz (2) of complexity being placed between order and disorder, we have arrived at a measure that is no longer open to the most pertinent criticisms, and whose construction is entirely transparent. Binder and Perry's requirement [22], that at least some classes of systems known from hierarchical analysis should be discernible, is satisfied to a somewhat weaker extent: whereas different

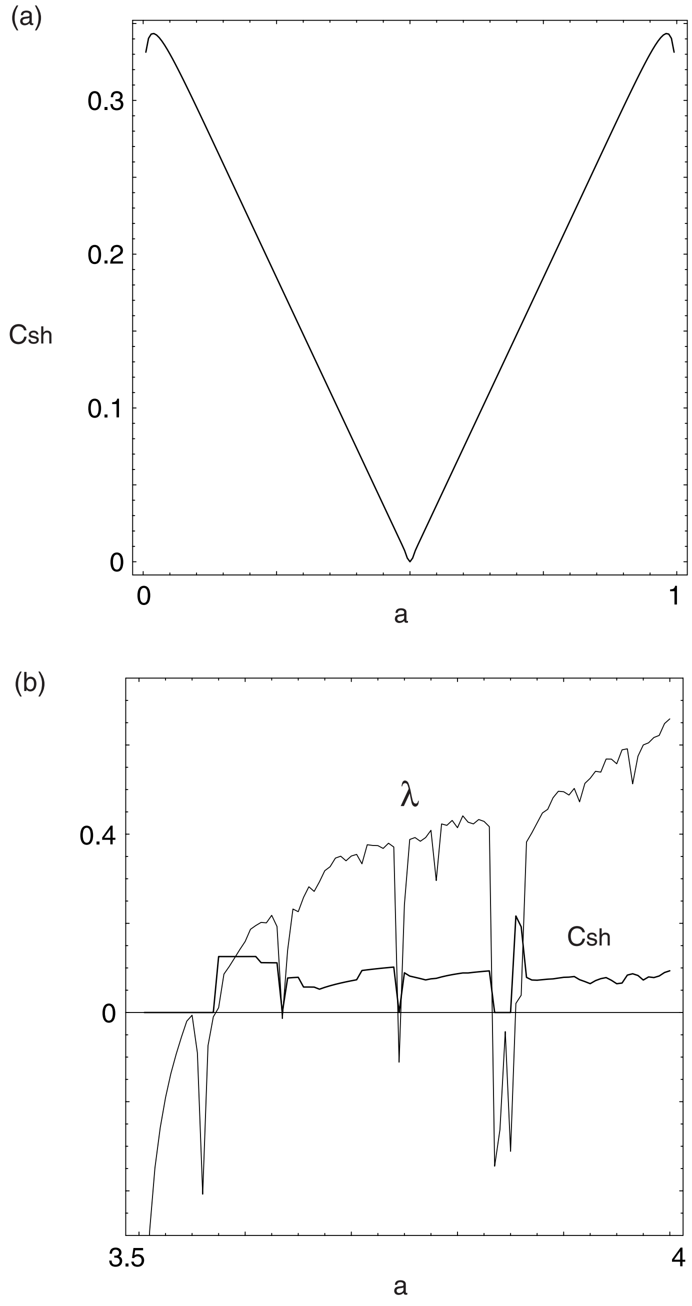

Figure 4. (a) Modified SDL complexity C_{Sh} (for $\alpha = \beta = 1$) of (a) the tent map family, as a function of the asymmetry a, and (b) for the logistic map family, as a function of the order parameter a (together with the Lyapunov exponent λ).

dynamical systems will indeed yield distinct complexity measure families, it is not obvious how this could be extended to properties of whole classes of systems. Using the simpler integrand (6), this was possible, in the sense that the highest measures for positive exponents were obtained for intermittent systems.

Whether a similar deeper significance can be attributed to the integrand (5), and how useful the measure could be for practical applications, remain to be seen. The linear dependence of the modified SDL complexity on the asymmetry, as observed for the tent map family, provides at least an interesting starting point. To become a convincing concept, also a class of systems should be identified that comprehensibly deserves a maximal SDL measure of complexity.

Although our analysis mainly refers to one-dimensional systems, the results obtained extend to more general cases. Entropy functions emerge also for higher dimensional systems (e.g., even from time series [2]), and the dependence of our complexity measure on non-hyperbolicities has been shown to be minimal. Composed systems may be represented as mixtures in the thermodynamical sense, or as multidimensional entropy functions, possibly reducible to product form. Therefore, we expect the higher dimensional situation, and the spatially extended case to a lesser extent, to not differ substantially from the one described here.

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