Entanglement, Disentanglement and Wigner Functions

2004 Phys. Scr. 69 166

(http://iopscience.iop.org/1402-4896/69/3/002)

Physica Scripta. Vol. 69, 166–169, 2004

Entanglement, Disentanglement and Wigner Functions

Yorick Hardy[1], Willi-Hans Steeb[1,*] and Ruedi Stoop[1,2]

[1] International School for Scientific Computing Rand Afrikaans University, P.O. Box 524, Auckland Park 2006, South Africa
[2] Institut für Neuroinformatics, ETHZ/UNIZH, Zürich, Switzerland

Received April 30, 2003, revised version received November 4, 2003; accepted November 6, 2003

PACS Ref: 03.65.Ud

Abstract

Entangled quantum states are an important component of quantum computing techniques such as quantum error-correction, dense coding and quantum teleportation. We describe how to generate fully entangled states in the Hilbert space $\boldsymbol{C}^N \otimes \boldsymbol{C}^N$ starting from a unitary matrix and show that they form an orthonormal basis in this space. Disentanglement is also discussed. Moreover we also calculate the Wigner function of fully entangled states.

Entanglement [1–7] is the characteristic trait of quantum mechanics which enforces its entire departure from classical lines of thought. It is nowadays viewed as a resource for certain tasks that can be performed faster or in a more secure way than classically. Einstein *et al.* [8] discussed entanglement for infinite-dimensional systems (position and momentum). Bohm [9] described the case for finite-dimensional systems.

We describe how to generate fully entangled states in the finite-dimensional Hilbert space $\boldsymbol{C}^N \otimes \boldsymbol{C}^N = \boldsymbol{C}^{N^2}$ starting from the $N \times N$ primary permutation matrix and show that they form an orthonormal basis in this space. We also describe how to disentangle the generalized Bell states using the GXOR-operator. Furthermore the Wigner operator for finite-dimensional systems is calculated for the Bell states.

We consider entanglement of pure states. For example in the Hilbert space $\boldsymbol{C}^4$ the Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle),$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

are fully entangled states and form an orthonormal basis in $\boldsymbol{C}^4$. Here $\{|0\rangle,\ |1\rangle\}$ is an arbitrary orthonormal basis in the Hilbert space $\boldsymbol{C}^2$. If we choose

$$|0\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$$

we obtain

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\Phi^-\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi}\cos(2\theta) \\ e^{i\phi}\sin(2\theta) \\ e^{i\phi}\sin(2\theta) \\ -\cos(2\theta) \end{pmatrix},$$

$$|\Psi^+\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{2i\phi}\sin(2\theta) \\ e^{i\phi}\cos(2\theta) \\ e^{i\phi}\cos(2\theta) \\ \sin(2\theta) \end{pmatrix}, \qquad |\Psi^-\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{pmatrix}.$$

If we choose $\phi = 0$ and $\theta = 0$ then $\{|0\rangle, |1\rangle\}$ is the standard basis in $\boldsymbol{C}^2$.

Consider the Hilbert space $\boldsymbol{C}^N$. Let

$$\{|\phi_k\rangle : k = 0, 1, \dots, N-1\} \tag{1}$$

be an orthonormal basis in $\boldsymbol{C}^N$. Thus $\langle\phi_j|\phi_k\rangle = \delta_{jk}$ and

$$\sum_{k=0}^{N-1} |\phi_k\rangle\langle\phi_k| = I_N \tag{2}$$

where I_N is the $N \times N$ unit matrix. The last relation is the completeness relation. Next we define the matrix

$$U := \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|. \tag{3}$$

Thus we can also write

$$U := \sum_{k=0}^{N-1} |\phi_{k \bmod N}\rangle\langle\phi_{k+1 \bmod N}|. \tag{4}$$

For example, in $\boldsymbol{C}^2$ we obtain the Pauli matrix σ_x if $|\phi_0\rangle = (1,0)^T$ and $|\phi_1\rangle = (0,1)^T$ and the Pauli matrix σ_z if $|\phi_0\rangle = 1/\sqrt{2}\,(1,1)^T$ and $|\phi_1\rangle = 1/\sqrt{2}\,(1,-1)^T$.

The set of matrices $\{U, U^2, \dots, U^N\}$ form a commutative group under matrix multiplication where $U^N = I_N$. We also have $\mathrm{tr}(U) = 0$ and $\det(U) = -1$ if N is even and $\det(U) = 1$ if N is odd. The eigenstates of the unitary matrix U satisfy

$$U|\theta_j\rangle = \exp(-i\theta_j)|\theta_j\rangle, \qquad j = 0, 1, \dots,\ N-1 \tag{5}$$

where $\theta_j := 2\pi j/N$. Thus the set of eigenvalues $\{\exp(-i\theta_j) : j = 0, 1, \dots, N-1\}$ form a commutative group under

* e-mail: WHS@NA.RAU.AC.ZA

multiplication. The group given above and this group are isomorphic. For $N \to \infty$ we have the Lie group $U(1)$.

For the standard basis in $\boldsymbol{C}^N$ the matrix U is given by

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6}$$

Then the set of matrices $\{U, U^2, \dots, U^N\}$ is a subgroup of the group of all $N \times N$ permutation matrices under matrix multiplication.

The normalized eigenvectors of U given by (3) are

$$|\theta_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(-\mathrm{i}2\pi jk)|\phi_k\rangle \tag{7}$$

where $j = 0, 1, \dots, N-1$. If we consider U given by (6) (i.e., the standard basis is selected), then we find the eigenvectors

$$\frac{1}{\sqrt{N}}(1, \exp(-\mathrm{i}2\pi j/N), \exp(-\mathrm{i}4\pi j/N), \dots,$$
$$\times \exp(-\mathrm{i}2\pi(N-1)j/N))^T.$$

Thus for the eigenvalue 1 of U we find the normalized eigenvector

$$\frac{1}{\sqrt{N}}(1, 1, \dots, 1)^T.$$

The eigenstates $\{|\theta_j\rangle\colon j = 0, 1, \dots, N-1\}$ and the orthonormal basis given above are connected by the discrete Fourier transform

$$|\theta_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(-\mathrm{i}k\theta_j)|\phi_k\rangle,$$

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp(\mathrm{i}k\theta_j)|\theta_j\rangle. \tag{8}$$

Next we introduce the two matrices

$$\hat{n} := \sum_{k=0}^{N-1} k|\phi_k\rangle\langle\phi_k|, \qquad \hat{\theta} := \sum_{j=0}^{N-1} \theta_j|\theta_j\rangle\langle\theta_j|. \tag{9}$$

We have $[U, \hat{\theta}] = 0$. The hermitian matrix $\hat{n}$ can be considered as the number operator with the eigenvalues $0, 1, 2, \dots, N-1$. The matrix $\hat{n}$ is diagonal in the standard basis. The matrix $\hat{\theta}$ is called the Pegg–Barnett phase operator [10]. We note that an outstanding problem in quantum mechanics is the search for a "proper" phase operator. A number of theories for such operators have been proposed, but most of them succumb to one or more of three shortcomings: (*i*) the operator is non-selfadjoint, (*ii*) no scheme for an experimental realization, (*iii*) the operator is operationally defined, leaving the questions open as to what observable the measurement apparatus really represents and what the conjugate observable is [11].

The matrix $\hat{n}$ has the properties

$$\exp(\pm\mathrm{i}\theta_j\hat{n})|\phi_k\rangle = \exp(\pm\mathrm{i}\theta_j k)|\phi_k\rangle,$$

$$\exp(\pm\mathrm{i}\theta_j\hat{n})|\theta_k\rangle = |\theta_{k\mp j \bmod N}\rangle. \tag{10}$$

The matrix $\hat{\theta}$ has the properties

$$\exp(\pm\mathrm{i}k\hat{\theta})|\theta_j\rangle = \exp(\pm\mathrm{i}k\theta_j)|\theta_j\rangle,$$

$$\exp(\pm\mathrm{i}k\hat{\theta})|\phi_m\rangle = |\phi_{m\mp k \bmod N}\rangle. \tag{11}$$

Note that the matrices $\hat{n}$ and $\hat{\theta}$ do not commute. Using these two matrices we introduce the matrices

$$\hat{V}_{jk} := \exp(\mathrm{i}\theta_j\hat{n})\exp(-\mathrm{i}k\hat{\theta}) \tag{12}$$

where $j, k = 0, 1, \dots, N-1$ and $V_{00} = I_N$. Obviously these N^2 matrices are unitary. Inserting (9), (10) and (11) into (12) we can write

$$\hat{V}_{jk} = \sum_{m=0}^{N-1} \exp(\mathrm{i}\theta_j m)|\phi_{m \bmod N}\rangle\langle\phi_{m-k \bmod N}|. \tag{13}$$

Let

$$|\Phi\rangle := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\phi_k\rangle \otimes |\phi_k\rangle \tag{14}$$

where $\otimes$ denotes the Kronecker product of matrices [12]. This state is independent of the chosen basis in each subsystem. Using this state we define the N^2 states

$$|\Phi_{jk}\rangle := (\hat{V}_{jk} \otimes I_N)|\Phi\rangle \equiv (\exp(\mathrm{i}\theta_j\hat{n})\exp(-\mathrm{i}k\hat{\theta}) \otimes I_N)|\Phi\rangle \tag{15}$$

where $|\Phi\rangle = |\Phi_{00}\rangle$. These states can also be written as

$$|\Phi_{jk}\rangle = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \mathrm{e}^{\mathrm{i}(2\pi/N)\ell j}|\phi_\ell\rangle \otimes |\phi_{\ell-k}\rangle. \tag{16}$$

The N^2 vectors $|\Phi_{jk}\rangle$ form an orthonormal basis in the Hilbert space $\boldsymbol{C}^{N^2}$, i.e., we have $\langle\Phi_{jk}|\Phi_{mn}\rangle = \delta_{jm}\delta_{kn}$ and [13]

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} |\Phi_{jk}\rangle\langle\Phi_{jk}| = I_N \otimes I_N = I_{N^2}.$$

The measure for entanglement for pure states $E(|\psi\rangle\langle\psi|)$ is defined as follows [2–7]

$$E(|\psi\rangle\langle\psi|) := S_{\dim(\mathcal{H}_1)}(\rho_{\mathcal{H}_1}) = S_{\dim(\mathcal{H}_2)}(\rho_{\mathcal{H}_2}) \tag{17}$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \boldsymbol{C}^N$ and the density operators are defined as

$$\rho_{\mathcal{H}_1} := \mathrm{Tr}_{\mathcal{H}_2}|\psi\rangle\langle\psi|, \qquad \rho_{\mathcal{H}_2} := \mathrm{Tr}_{\mathcal{H}_1}|\psi\rangle\langle\psi| \tag{18}$$

and $S_b(\rho) := -\mathrm{Tr}\rho\log_b\rho$. Tr denotes the trace and $\mathrm{Tr}_{\mathcal{H}_1}$ denotes the partial trace over $\mathcal{H}_1$. We use the base b for the logarithm $\log_b$. We have $0 \log_b 0 = 0$ and $1 \log_b 1 = 0$. Thus $0 \leq E \leq 1$. If $E = 1$ we call the pure state maximally entangled. If $E = 0$, the pure state is not entangled. We note that

$$S_b(\rho) = -\sum_{j=1}^{k} \lambda_j \log_b \lambda_j$$

where $\{\lambda_j : j = 1, \ldots, k\}$ are the eigenvalues of ρ and ρ is a linear operator on a k dimensional Hilbert space.

Using this definition for entanglement of pure states we find that the states $|\Phi_{jk}\rangle$ are maximally entangled. We obtain the projection matrix

$$|\Phi_{jk}\rangle\langle\Phi_{jk}| = \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \exp(i\theta_j(l-m)) |\phi_{l+k}\rangle \times \langle\phi_{m+k}| \otimes |\phi_l\rangle\langle\phi_m|. \tag{19}$$

Taking the partial trace over the first system yields

$$\rho_2 := \mathrm{Tr}_1(|\Phi_{jk}\rangle\langle\Phi_{jk}|) = \frac{1}{N} \sum_{l=0}^{N-1} |\phi_l\rangle\langle\phi_l|.$$

Thus $S_N(\rho_2) = 1$. The state $|\Phi\rangle$ given by (14) is a maximally entangled state. Thus the other states are obtained by a local unitary transformation (15) from the maximally entangled one.

For $N = 2$ we find the Bell states given above. For $N = 3$ we obtain the 9 states

$$|\Phi_{00}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle),$$

$$|\Phi_{10}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + e^{i(2\pi/3)}|11\rangle - e^{i(\pi/3)}|22\rangle),$$

$$|\Phi_{20}\rangle = \frac{1}{\sqrt{3}}(|00\rangle - e^{i(\pi/3)}|11\rangle + e^{i(2\pi/3)}|22\rangle),$$

$$|\Phi_{01}\rangle = \frac{1}{\sqrt{3}}(|10\rangle + |21\rangle + |02\rangle),$$

$$|\Phi_{02}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |20\rangle),$$

$$|\Phi_{11}\rangle = \frac{1}{\sqrt{3}}(|02\rangle + e^{i(2\pi/3)}|10\rangle - e^{i(\pi/3)}|21\rangle),$$

$$|\Phi_{21}\rangle = \frac{1}{\sqrt{3}}(|02\rangle - e^{i(\pi/3)}|10\rangle + e^{i(2\pi/3)}|21\rangle),$$

$$|\Phi_{12}\rangle = \frac{1}{\sqrt{3}}(|01\rangle + e^{i(2\pi/3)}|12\rangle - e^{i(\pi/3)}|20\rangle),$$

$$|\Phi_{22}\rangle = \frac{1}{\sqrt{3}}(|01\rangle - e^{i(\pi/3)}|12\rangle + e^{i(2\pi/3)}|20\rangle)$$

where $|00\rangle = |0\rangle \otimes |0\rangle$ etc. and $|0\rangle$, $|1\rangle$, $|2\rangle$ denote the standard basis in $\boldsymbol{C}^3$.

Given an entangled state in $\boldsymbol{C}^N \otimes \boldsymbol{C}^N$ it is important to know if it can be distilled, i.e., r copies of it can be transformed by local operations and classical communication into s copies of $|\Phi\rangle$. State distillability, or useful quantum correlations, offer an alternative way of analyzing quantum nonlocality. All bipartite entangled pure states can be reversibly transformed using local operations and classical communication into $|\Phi\rangle$ (in the so-called asymptotic regime).

The set of the N^2 projection matrices

$$\{X_{jk} = |\Phi_{jk}\rangle\langle\Phi_{jk}| : j, k = 0, 1, \ldots, N-1\}$$

describe the generalized Bell measurement. An application of the Bell measurement is in teleportation. For any matrix

$$\hat{O} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} O_{mn} |\phi_m\rangle\langle\phi_n|$$

we have

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \hat{V}_{jk} \hat{O} \hat{V}_{jk}^* = N(\mathrm{tr}\hat{O}) I_N.$$

To disentangle the generalized Bell states the generalized XOR-gate (GXOR-gate) can be used. The GXOR-gate is defined as [14]

$$U_{\mathrm{GXOR}} |m\rangle \otimes |n\rangle := |m\rangle \otimes |m \ominus n\rangle$$

where $m, n = 0, 1, \ldots, N-1$ and

$$m \ominus n := m - n \text{ modulo } N.$$

The operator U_{GXOR} is unitary and hermitian and therefore $U_{\mathrm{GXOR}} = U_{\mathrm{GXOR}}^{-1}$. Furthermore $m \ominus n = 0$ modulo N if and only if $m = n$. For $N = 2$ we obtain the XOR-gate. For $N = 3$ we find the matrix representation

$$U_{\mathrm{GXOR}} = 1 \oplus U_{NOT}^{(2)} \oplus U_{NOT}^{(2)} \oplus 1 \oplus U_{NOT}^{(3)}$$

where $\oplus$ denotes the direct sum [12] and

$$U_{NOT}^{(2)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U_{NOT}^{(3)} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus U_{GXOR} is a permutation matrix. For example for $N = 3$ a fully entangled state is given by (generalized Bell state)

$$|\Phi_{00}\rangle = \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle).$$

Then

$$U_{\mathrm{GXOR}} |\Phi_{00}\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \otimes |0\rangle.$$

The Wigner operator and Wigner function were originally introduced for infinite dimensional systems. For finite dimensional Hilbert spaces with dimension $N \geq 3$ prime the Wigner operator can be defined as [15,16]

$$\hat{W}(q, p) := \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \delta_{2q, r+s} \exp\left(i \frac{2\pi}{N} p(r-s)\right) |r\rangle\langle s|$$

where $q,\ p = 0, 1, \ldots, N-1$. The calculations are done modulo N, i.e., $r+s$ stands for $(r+s)$ modulo N, etc. The (q,p) pairs constitute the discrete phase space. For $N = 3$ the matrix representation of $\hat{W}$ is

$$\hat{W}(q,p) = \begin{pmatrix} \delta_{2q,0} & \delta_{2q,1}\, e^{-i2\pi p/3} & \delta_{2q,2}\, e^{-i4\pi p/3} \\ \delta_{2q,1}\, e^{i2\pi p/3} & \delta_{2q,2} & \delta_{2q,3}\, e^{-i2\pi p/3} \\ \delta_{2q,2}\, e^{i4\pi p/3} & \delta_{2q,3}\, e^{i2\pi p/3} & \delta_{2q,4} \end{pmatrix}$$

where $\delta_{2q,3} \equiv \delta_{2q,0}$ and $\delta_{2q,4} \equiv \delta_{2q,1}$. For a state described by a density matrix ρ the Wigner function is

$$W(q,p) = \frac{1}{N}\operatorname{tr}(\rho \hat{W}).$$

Wigner functions defined in this way obey analogous properties to those defined on infinite-dimensional Hilbert spaces. The marginal distributions of the functions

$$P_q(q) = \sum_{p=0}^{N-1} W(q,p), \qquad P_p(p) = \sum_{q=0}^{N-1} W(q,p)$$

describe the statistics of measurements of observables $\hat{q}$ and $\hat{p}$, respectively. For a bipartite system with subsystems 1 and 2 described by the joint density matrix $\rho^{(12)}$ we define

$$W(q_1,p_1,q_2,p_2) = \frac{1}{N^2}\operatorname{tr}(\rho^{(12)}\hat{W}_1(q_1,p_1) \otimes \hat{W}_2(q_2,p_2)).$$

Wigner functions describing a subsystem are obtained by summing the joint Wigner function in the corresponding set of the respective variables, i.e.,

$$W(q_1,p_1) = \sum_{q_2=0}^{N-1}\sum_{p_2=0}^{N-1} W(q_1,p_1,q_2,p_2),$$

$$W(q_2,p_2) = \sum_{q_1=0}^{N-1}\sum_{p_1=0}^{N-1} W(q_1,p_1,q_2,p_2).$$

For the generalized Bell state $|\Phi_{00}\rangle$ we find

$$W_{\Phi_{00}}(q_1,p_1,q_2,p_2) = \frac{1}{N^2}\delta_{q_1,q_2}\delta_{p_1,-p_2}.$$

Thus $W_{\Phi_{00}}$ is nonzero if and only if $q_1 - q_2 = 0$ and $p_1 + p_2 = 0$. The Wigner function given above shows the connection with the EPR-state for continuous-variable teleportation [7,8]

$$\delta(q_1 - q_2) \otimes \delta(p_1 + p_2).$$

We have shown, starting from the circulant matrix U, how to construct fully entangled states in finite dimensional Hilbert spaces applying the Pegg–Barnett operator. To prove that the states are fully entangled we have calculated the entanglement measure $E(|\psi\rangle\langle\psi|)$. To disentangle the Bell states, i.e. produce product states, we applied the U_{GXOR} operator. Thus the U_{GXOR} operator provides another technique to generate fully entangled states. Finally we have calculated the Wigner operators and Wigner functions for the generalized Bell states. The analogy with the EPR-state has been established.

References

1. Schrödinger, E., Proc. Cambridge Philosophical Soc. **32**, 446 (1935).
2. Preskill, J., "The Theory of Quantum Information and Quantum Computation," California Inst. of Tech., Pasadena, CA, (2000), http://www.theory.caltech.edu/~preskill/ph229
3. Hardy, Y. and Steeb, W.-H., "Classical and Quantum Computing with C++ and Java Simulations," (Birkhäuser, Basel, 2001).
4. Nielsen, M. A. and Chuang, I. L., "Quantum Computation and Quantum Information," (Cambridge University Press, Cambridge, 2000).
5. Steeb, W.-H. and Hardy, Y., Int. J. Theor. Phys. **39**, 2765 (2000).
6. Steeb, W.-H. and Hardy, Y., Z. Naturforsch. **57a**, 400 (2002).
7. Steeb, W.-H. and Hardy, Y., "Problems and Solutions in Quantum Computing and Quantum Information," (World Scientific, Singapore, 2004).
8. Einstein, A., Podolsky, B. and Rosen, N., Phys. Rev. **47**, 777 (1935).
9. Bohm, D., "Quantum Theory of the Measurement Process", in: "Quantum Theory," (Prentice-Hall, Englewood Cliffs, N.J., 1951) p.583.
10. Pegg, D. T. and Barnett, S. M., J. Mod. Opt. **44**, 225 (1997).
11. Trifonov, A. *et al.*, J. Opt. B: Quantum Semiclass. Opt. **2**, 105 (2000).
12. Steeb, W.-H., "Matrix Calculus and Kronecker Product with Applications and C++ Programs", (World Scientific, Singapore 1997).
13. Ban, M., J. Phys. A: Math. Gen. **35**, L193 (2002).
14. Albers, G., Delgado, A., Gisin, N., and Jex I., J. Phys. A: Math. Gen. **34**, 8821 (2001).
15. Wootters, W. K., Ann. Phys. (N.Y.) **176**, 1 (1987).
16. Vourdas, A., J. Phys. A: Math. Gen. **29**, 4275 (1996).